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Yang-Mills correlation functions from integrable spin chains

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Abstract: The relation between the dilatation operator of $\mathcal{N}=4$ Yang-Mills theory and integrable spin chains makes it possible to compute the one-loop anomalous dimensions of all operators in the theory. In this paper we show how to apply the technology of integrable spin chains to the calculation of Yang-Mills correlation functions by expressing them in terms of matrix elements of spin operators on the corresponding spin chain. We illustrate this method with several examples in the SU(2) sector described by the $\mathrm{XXX}_{1/2}$ chain.

Keywords: AdS-CFT and dS-CFT Correspondence, Integrable Field Theories.

Contents

1. Introduction

Recently there has been considerable progress in studying the AdS/CFT correspondence beyond the BPS and near-BPS limits (see [1] for a review). Integrability on both sides of the correspondence plays an important role in these quantitative checks. On the gauge theory side it has been shown that the planar one-loop dilatation operator can be identified with the hamiltonian of an integrable spin chain [2]. The vector space on each site of the spin chain depends on the particular sector one is interested in. For instance for scalar operators the dilatation operator can be described by the SO(6) spin chain in the vector representation [2], while an $SU(2, 2|4)$ spin chain in the singleton representation describes the full $\mathcal{N} = 4$ Yang-Mills planar one-loop dilatation operator [3]. The Bethe ansatz allows for finding (though implicitly) the exact planar one-loop anomalous dimensions of arbitrary local operators in gauge theory.

On the string theory side of the correspondence, a different integrable structure has been found in the worldsheet theory [4] and was related to the gauge theory spin chain description in [5]. Nevertheless, finding the spectrum of string theory in AdS still remains a challenge. String theory becomes tractable in the semiclassical regime, when the quantum numbers of the string states become large [6, 7]. The energies of certain semiclassical soliton solutions with two large angular momenta were shown to match the anomalous dimensions of operators with large quantum numbers computed to one loop using the XXX spin chain in [8] and to two loops using the elliptic spin chain in [9]. In this context, the higher local

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conserved charges on both sides have been matched in [10]. Moreover, direct agreement has been shown between the continuum limit of the dilatation operator in the coherent state basis and the hamiltonian of the sigma model [11] without needing the details of the semiclassical solutions.

Less symmetric spin chains were also analyzed in relation to gauge theories. The XXZ chain with periodic boundary conditions was related to pure $\mathcal{N}=2$ SYM in [12], while the same chain but with twisted boundary conditions was related in [13] to the q-deformation of the $\mathcal{N}=4$ SYM. The same gauge theory was shown in [14] to be related to a limit of a multi-parametric spin chain which also describes certain non-supersymmetric theories. For related work on integrability in gauge and string theories see [15]–[60].

So far these studies have been limited to finding the eigenvalues of the dilatation operator. In this paper we show how to use the technology of integrable spin chains to find three-point functions of arbitrary scalar operators in $\mathcal{N}=4$ Yang-Mills and some of its deformations.[1] Even though it is relatively easy to compute three-point functions of operators at tree level, it becomes increasingly laborious to do so at higher loops. In fact, even the computation of the tree level three-point function coefficient of operators with definite one-loop scaling dimension is quite hard, because these operators are combinations, with occasionally rather complicated coefficients, of the natural tree level operators. The spin chain conveniently identifies a basis of operators with definite scaling dimension and leads to an algorithm for the computation of the leading three-point function coefficient. This coefficient turns out to be related to the expectation value of certain spin chain operators, a subject which has been extensively studied (see [62] and references therein). Loop corrections to the three-point function coefficient may be expressed as the products of expectation values of operators similar to the spin chain hamiltonian.

In section 2 we construct the map between gauge theory and spin chain correlation functions, emphasizing the spin chain realization of arbitrary scalar operators. In section 3 and section 4 we review the relevant details of the algebraic Bethe ansatz and spin chain correlation functions. We proceed in section 5 to describe several examples.

At one loop the spin chain description of various sectors of the gauge theory is not restricted to operators with large quantum numbers. The planar perturbative correlation functions of these operators are related to the tree-level scattering amplitudes of string theory in highly curved AdS. The limit in which one of the R-charges is large is similar to a plane wave limit. In this regime the corresponding string amplitudes for massless states have been computed in [63]. A naive extension of this calculation to massive states leads to an apparent puzzle: the three-point function involving one massive string state and two massless states vanishes identically. However, one of the examples we will discuss in section 5 leads to a nonvanishing result. Though we will not attempt it here, it would be interesting to clarify the relation between the correlation functions we discuss here and string theory computations, perhaps along the lines of [64].

[1] An earlier attempt in this direction [61] related some of their building blocks to certain open spin chains. We depart from this standpoint and use only closed spin chains, which are perhaps more natural from a string theory perspective.

2. $\mathcal{N} = 4$ Yang-Mills and spin chains: the operator map

The computation of planar one-loop anomalous dimensions of scalar operators in arbitrary representations of the R-symmetry group is conveniently encoded in the diagonalization of the hamiltonian of the Heisenberg spin chain with SO(6) symmetry [2],

$$\Delta = \frac{\lambda}{16\pi^2} \sum_{i=1}^{L} K_{i,i+1} + 2I_{i,i+1} - 2P_{i,i+1} \,, \tag{2.1}$$

where P, K and I are, respectively, the permutation, trace and identity operators acting on the tensor product of nearest neighbor spins i and $i+1$ and L is the length of the chain.

The spin chain description also provides a tool for the computation of various numerical coefficients appearing in correlation functions. Such techniques are particularly efficient in situations when the position dependence of the correlation function is known. The three-point functions of operators with definite conformal weights fall in this category. For operators of conformal dimension Δ_I, Δ_M and Δ_K the position dependence on the plane is fixed by (super)conformal invariance to

$$\langle \mathcal{O}^I(x)\, \mathcal{O}^M(y)\, \mathcal{O}^K(z) \rangle = \frac{C^{IMK}(\lambda)}{|x-y|^{\Delta_I+\Delta_M-\Delta_K} |x-z|^{\Delta_I-\Delta_M+\Delta_K} |y-z|^{-\Delta_I+\Delta_M+\Delta_K}} \,. \tag{2.2}$$

The OPE coefficients $C^{IMK}(\lambda)$ are in general a series in the 't Hooft parameter

$$C^{IMK}(\lambda) = c_0^{IMK}(1 + \lambda c_1^{IMK} + \mathcal{O}(\lambda^2)) \,. \tag{2.3}$$

The one-loop correction c_1^{IMK} was partly analyzed in [61]. Before discussing this correction however, it is important to have an efficient scheme for the computation of the leading coefficient c_0^{IMK}.

It turns out that the leading OPE coefficients have a simple description: since they are given by free field contractions, they can be identified with the matrix element of one of the three operators in (2.2) with the in- and out-states corresponding to the other two operators

$$c_0^{IMK} = \langle I|\, \hat{O}^M\, |J\rangle = \langle M|\, \hat{O}^I\, |J\rangle = \text{etc} \,. \tag{2.4}$$

Here $\hat{O}^M$ is the representation of $\mathcal{O}^M$ on the spin chain. There is no canonical way to choose which operators in (2.2) correspond to the in- and out-states. As we will see later it is useful to use this freedom to simplify the calculations. For example, operators with a small number of fields have simple expressions in terms of the basic spin chain operators and it is usually convenient to put them in the middle. We reach the conclusion that to compute c_0^{IMJ} we should compute the transition induced by some arbitrary operator between two eigenvectors of scale transformations, in some sector closed under RG flow.

An alternative reasoning leads to the conclusion that the leading OPE coefficients can be expressed in terms of the convolution of three scalar products of spin chain states. Indeed, given three spin chains of arbitrary lengths in some real representation it is possible to define an object similar to a three-string vertex which maps a tensor product of two

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states into a third one while taking into account planar free field contractions. In the case of extremal three-point functions this construction can be easily extended to spin chains in complex representations.

The advantage of this realization of c_0 is that it also provides a relatively simple way of computing the subleading coefficients $c_0 c_1$ and more generally the coefficients of the logarithmic corrections to the tree level three-point functions, in terms of the expectation values of certain local and nonlocal operators. With a similar goal, the authors of [61] introduced a map describing the splitting of a closed spin chain into two open spin chains. The construction we will describe at the end of this section bypasses this step by employing only the spin chains describing the operators whose three-point function we are interested in computing.

2.1 The SU(2) sector

The holomorphic operators are the easiest ones to analyze in the context of our discussion. The reason is that the dependence of the correlation function on the in- and out-states becomes easy to track due to R-charge conservation. This becomes particularly easy for operators belonging to an SU(2) sector of the theory,

$$\mathcal{O}^{i_1 \dots i_n} = \mathrm{Tr}[\phi^{i_1} \dots \phi^{i_n}]\,, \tag{2.5}$$

where $i_k = 1, 2$ and ϕ^i are holomorphic fields.

The dilatation operator in this sector is a restriction of (2.1) and acts on these operators as the negative of the hamiltonian of the SU(2) XXX Heisenberg spin chain,

$$H = \sum_{i=1}^{L} H_{i,i+1} \qquad H_{i,i+1} = 1 - P_{i,i+1} = -(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \sigma_i^z \sigma_{i+1}^z)\,, \tag{2.6}$$

with periodic boundary conditions, where $\sigma_i^\pm = \frac{1}{\sqrt{2}}(\sigma_i^1 \pm i\sigma_i^2)$ and σ_i^z are the Pauli matrices acting at the site i. The eigenvectors of H can be explicitly computed for short operators, or for very long operators, i.e. $L \to \infty$. In either case they are linear combinations of (2.5) with coefficients determined by the Bethe equations, which we will review in section 3.

Since the operators (2.5) are linearly related to the eigenvectors of (2.6) which are special configurations of spins on a chain, it follows that to compute the OPE coefficients we have to find a representation of some arbitrary operator in terms of spin chain variables. It is not hard to see that there is a qualitative difference between operators with the same number of holomorphic and antiholomorphic fields and those in which their number is different. For the first kind of operators, ordinary Feynman diagrams imply that the result of the contraction of the antiholomorphic fields with fields in the in-state is an operator with the same number of fields as the original operator and thus it is described by a state in the original Hilbert space. The operators of the second kind lead to a change in the number of fields and thus the resulting state belongs to the space of states of a chain of a different length. We will treat the two cases separately.

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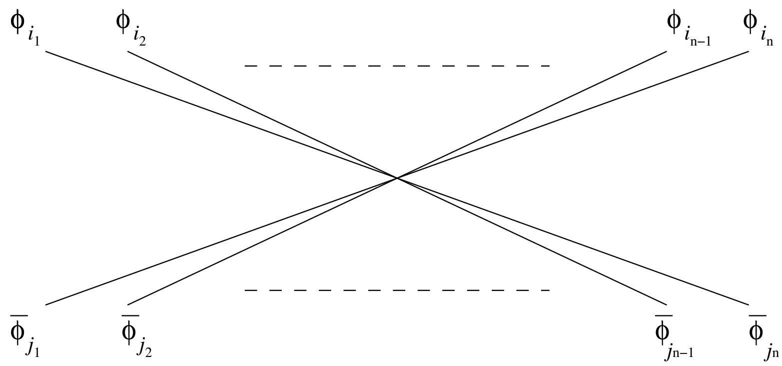

Figure 1: Graphic representation of the operator (2.7).

Operators with the same number of ϕ^i and $\bar{\phi}_j$. It is quite easy to find the action of such an operator on the eigenvectors of (2.6) (or, more generally, on a holomorphic operator) by analyzing the Feynman rules of the gauge theory. Since we are interested in the planar limit of the gauge theory, let us consider the operator

$$\mathcal{O}^{i_1..i_n}_{j_1..j_n} = \mathrm{Tr}[\phi^{i_n}\ldots\phi^{i_1}\bar{\phi}_{j_1}\ldots\bar{\phi}_{j_n}]\,, \tag{2.7}$$

which is schematically represented in figure 1 using the cyclicity of the trace.

We see that, at the planar level, this operator takes n ordered neighboring fields with SU(2) indices $(j_1,\ldots,j_n)$ and transforms them into n ordered neighboring fields with indices $(i_1,\ldots,i_n)$. If (2.7) acts on an operator which does not contain the sequence of indices $(j_1,\ldots,j_n)$ then the result vanishes identically. On a case by case basis the spin chain representation of the operator (2.7) can be written quite explicitly.

From figure 1 it is relatively clear how to do this: due to the cyclicity of the trace we see that the k-th field in (2.7) is transformed into the $2n-k+1$-th field. Depending on which one of ϕ^1, ϕ^2 or their antiholomorphic versions these fields are, this transition is realized on the spin chain by one of the SU(2) generators. For example, using conventions compatible with (2.6) and section 3, we find that

$$\mathrm{Tr}[\phi^1\bar{\phi}_1] \quad\longleftrightarrow\quad \sum_i \frac{1}{2}(1+\sigma^z_i) \qquad \mathrm{Tr}[\phi^2\bar{\phi}_2] \quad\longleftrightarrow\quad \sum_i \frac{1}{2}(1-\sigma^z_i)\,, \tag{2.8}$$

which are the number operators for ϕ^1 and ϕ^2, respectively. A more involved example is the one containing all operators with, say, n fields:

$$\begin{aligned}\mathrm{Tr}&\left[\prod_{j=1}^{n}\left[w^1_j\phi^1+w^2_j\phi^2\right]\prod_{i=1}^{n}\left[v^1_{n+1-i}\bar{\phi}_1+v^2_{n+1-i}\bar{\phi}_2\right]\right]\longleftrightarrow\\ &\longleftrightarrow\sum_{i=1}^{L}\prod_{j=i}^{i+n}\left[u^{21}_i\sigma^-_i+u^{12}_i\sigma^+_i+\frac{u^{11}_i}{2}(1+\sigma^z_i)+\frac{u^{22}_i}{2}(1-\sigma^z_i)\right]\,,\end{aligned} \tag{2.9}$$

with the coefficients u related to v and w by

$$u^{mn}_i = w^m_i v^n_i \tag{2.10}$$

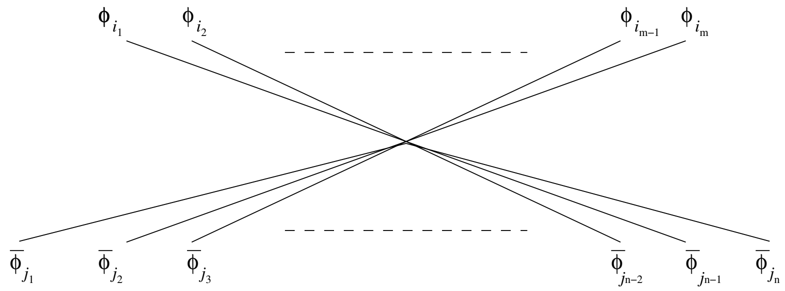

Figure 2: Graphic representation of the operator (2.11).

and the site labels defined cyclically $i \simeq i + L$. Computing the matrix element of this operator between some fixed in- and out-states leads to the generating function for the tree level coefficients c_0^{IMJ} with fixed I and J (corresponding to the in- and out-states) and M of fixed size, but arbitrary SU(2) structure.

Operators with different number of ϕ^i and $\bar{\phi}_j$. As discussed before, these operators are qualitatively different from the operators with the same number of holomorphic and antiholomorphic fields, because their action on the spin chain changes its length. Indeed, the operator

$$\mathcal{O}^{l_1\ldots l_m}_{j_1\ldots j_n} = \mathrm{Tr}[\phi^{l_m}\ldots\phi^{l_1}\bar{\phi}_{j_1}\ldots\bar{\phi}_{j_n}] \tag{2.11}$$

schematically represented in figure 2, takes n ordered neighboring fields $(j_1,\ldots,j_n)$ and replaces them with m ordered neighboring fields $(l_1,\ldots,l_m)$ changing therefore the number of fields. Because of this fact, such operators cannot have an expression similar to (2.9), but they also must contain a map between the spaces of states of spin chains of different lengths.

In the context of our discussion it is relatively easy to construct this map based on the behavior of the operators (2.11). The idea is to project the in state on states which contain the sequence $(j_1,\ldots,j_n)$, replace this sequence with $(l_1,\ldots,l_m)$ and sum over all such states. The result is that we have to compute the expectation value of

$$\mathcal{O}^{l_1\ldots l_m}_{j_1\ldots j_n} \longleftrightarrow \sum_{\substack{k=0\\ \text{all } i_s=1,2}}^{L-1} |\phi^{i_1}..\,\phi^{i_k}\,(\phi^{l_1}..\,\phi^{l_m})\,\phi^{i_{k+n}}..\,\phi^{i_L}\rangle\langle\bar{\phi}_{i_1}..\,\bar{\phi}_{i_k}\,(\bar{\phi}_{j_1}..\,\bar{\phi}_{j_n})\,\bar{\phi}_{i_{k+n+1}}..\,\bar{\phi}_{i_L}| \tag{2.12}$$

where L is the length of the chain corresponding to the in-state, the sum is over all $i_1 = 1,2$; $\ldots i_{k-1} = 1,2$; $i_{k+n+1} = 1,2$; $\ldots i_N = 1,2$; and for $k \geq L-n$ the states are defined using the cyclicity of the trace. Each of the two states appearing in every term in the sum above can be obtained by acting with SU(2) generators on the out and in ground states of the spin chains of length L and $L+m-n$, respectively, and must be unit normalized. Here $|\phi^{i_1}\ldots\phi^{i_n}\rangle = |\phi^{i_1}\rangle\otimes\ldots\otimes|\phi^{i_n}\rangle$ where $|\phi^1\rangle = |0\rangle$ and $|\phi^2\rangle = \sigma^-|0\rangle$.

It is worth mentioning that the representation (2.12) extends to operators with the same number of holomorphic and antiholomorphic fields (2.7) by just picking $m = n$. Depending on the context, it may be that one of the two representations (2.9) or (2.12) is more convenient. Clearly, (2.12) leads to OPE coefficients involving only scalar products which may be better-suited for a general analysis.

An additional useful point is that the equation (2.12) can be conveniently written in terms of the shift operator along the chain, $\tau(0)$,

$$\tau(0)|\phi^{i_1}\phi^{i_2}\dots\phi^{i_L}\rangle = |\phi^{i_L}\phi^{i_1}\dots\phi^{i_{L-1}}\rangle\,. \tag{2.13}$$

For the purpose of computing correlation functions of gauge invariant operators (represented by eigenvectors of the spin chain transfer matrix) this is particularly useful because these operators are invariant under such shifts. Thus, using

$$\mathcal{O}^{l_1..l_m}_{j_1..j_n} \quad\longleftrightarrow\quad \sum_{\substack{k=0\\ \text{all }\ i_s=1,2}}^{L-1} \tau(0)^k|(\phi^{l_1}..\phi^{l_m})\,\phi^{i_{n+1}}..\phi^{i_L}\rangle\langle(\bar\phi_{j_1}..\bar\phi_{j_n})\,\bar\phi_{i_{n+1}}..\bar\phi_{i_L}|\tau(0)^k \tag{2.14}$$

the summation over k leads only to multiplication by the length of the chain.

2.2 The SO(6) sector

The discussion in the previous subsection extends, with some modification, to operators in the SO(6) sector, i.e. all scalar operators in the theory. The main difference is that, unlike in the SU(2) case, the fields transform in a real representation and therefore all fields in the operator whose matrix element we want to compute can have a nontrivial contraction with the fields in the in- and out-states. The only constraint comes from its trace structure.

As before, given an in- and out-state with k_I and k_J fields respectively, an operator with k_M fields will have a nontrivial matrix element between the given in- and out-states if

$$k_I = m+n \qquad k_J = m+p \qquad k_M = n+p\,. \tag{2.15}$$

Similarly to the SU(2) case, it is useful to make a distinction between the cases $n=p$ (or $k_I = k_J$) and $n\neq p$ (or $k_I\neq k_J$). In the first case we will have again two apparently different expressions $\hat O^M$ which may be useful in different contexts.

If $n=p$ the operator $\hat O^M$ can be schematically represented as in figure 1, except that now we have to sum over all possible ways of choosing n neighboring fields to be contracted with the in-state. The operator

$$\mathcal{O}^M = \mathrm{Tr}[\phi^{i_1}\dots\phi^{i_{2n}}] \tag{2.16}$$

is represented on a spin chain by

$$\sum_{j=1}^{L}\sum_{k=1}^{2n}(E_j)^{i_{2n+1-k}}_{i_k}\dots(E_{j+n-1})^{i_{k+n}}_{i_{k+n-1}} = \sum_{i=1}^{L}\tau(0)^i\sum_{k=1}^{2n}(E_1)^{i_{2n+1-k}}_{i_k}\dots(E_n)^{i_{k+n}}_{i_{k+n-1}}\tau(0)^{-i}\,, \tag{2.17}$$

where $(E_n)^j_i$ are the generators of the general linear group

$$((E_n)^i_j)^l_k = \delta^i_k\delta^l_j \tag{2.18}$$

with the index n labeling the site and all indices are defined cyclically

$$i + L \simeq i \qquad i_{k+2n} \simeq i_k \,. \tag{2.19}$$

If $n \neq p$ the operator $\hat{O}^M$ can be schematically represented as in figure 2, with the same provision that we have to sum over all possibilities of choosing n neighboring fields to be contracted with the fields of the in-state. The analog of equation (2.14) is

$$\mathcal{O}^{n+p} \longleftrightarrow \sum_{\substack{k=0 \\ \text{all } \ i_s=1\dots6}}^{L-1} \tau(0)^k |(\phi^{l_{s+1}} .. \phi^{l_{s+p}})\phi^{i_{k+n+1}} .. \phi^{i_L}\rangle\langle(\phi^{l_{s+p+1}} .. \phi^{l_{s+n+p}})\phi^{i_{k+n+1}} .. \phi^{i_L}|\tau(0)^k \tag{2.20}$$

where, as before, all indices are cyclically defined

$$i + L \simeq i \qquad l_{s+n+p} \simeq l_s \,. \tag{2.21}$$

It is certainly possible to extend this construction to operators containing fermions. Perhaps the simplest such sector is the one containing two scalars and one fermion (the gaugino in an $\mathcal{N} = 1$ language), which has a manifest SU(1|2) symmetry. The extension to the full theory (the full SU(4|4) spin chain) is complicated by the fact that the Feynman diagram interpretation of the dilatation operator is not immediately obvious. We will not attempt to discuss this here and leave it for future work.

An important observation is that the technique used to construct the matrix elements of operators with $n \neq p$ and different number of holomorphic and antiholomorphic fields can be also used to describe the mixing of single- and double-trace operators. Indeed, equations analogous to (2.12) and (2.20) represent maps between the space of states of a single chain and the tensor product of the spaces of states of two different chains of different lengths. Following the logic outlined before, the leading OPE coefficient describing the transition between a single-trace and a double-trace operator is given by the matrix element of

$$\sum_{\substack{s=0 \\ \text{all } \ i_n=1\dots6}}^{L} |\phi^{i_{s+L_1+1}} \dots \phi^{i_{s+L_2}}\rangle_{\text{mod } L_2} \otimes |\phi^{i_{s+1}} \dots \phi^{i_{s+L_1}}\rangle_{\text{mod } L_1} \langle\bar{\phi}_{i_1} \dots (\bar{\phi}_{i_{s+1}} \dots \bar{\phi}_{i_{s+L_1}}) \dots \bar{\phi}_{i_L}| \tag{2.22}$$

between an eigenstate of a chain of length L and the tensor product of two states belonging to chains of lengths L_1 and $L_2 = L - L_1$, respectively. In the same spirit, the leading OPE coefficient for three operators described by spin chains in a real representation (such as the vector representation of SO(6)) is given by the expectation value of

$$\begin{aligned} &\sum_{s_1=0}^{L_1-1} \sum_{s_2=0}^{L_2-1} \sum_{s_3=0}^{L_3-1} \sum_{\text{all } i,j} \tau(0)^{s_1}_{L_1} |(\phi^{i_1} \dots \phi^{i_m})(\phi^{i_{m+1}} \dots \phi^{i_{L_1}})\rangle \otimes \\ &\otimes \tau(0)^{s_2}_{L_2} |(\phi^{i_1} \dots \phi^{i_m})(\phi^{j_{m+1}} \dots \phi^{j_{L_2}})\rangle\langle(\phi_{i_{m+1}} \dots \phi_{i_{L_1}})(\phi_{j_{m+1}} \dots \phi_{j_{L_2}})|\tau(0)^{s_3}_{L_3} \end{aligned} \tag{2.23}$$

between an eigenstate of a chain of length L_1 and the tensor product of two states belonging to chains of lengths L_2 and L_3 obeying a relation similar to (2.15).

2.3 The one-loop correction

The form given by (2.23) from the tree level OPE coefficient lends itself to higher order calculations as well. For the sake of simplicity we will exemplify this at the one-loop level, but the extension to higher loops is conceptually relatively clear.

There are two different types of Feynman diagrams contributing to a one-loop three-point function: 1) the loop involves fields belonging to only two of the three operators and 2) the four fields belong to three operators. Since only F-terms contribute nontrivially to three-point functions [65], in both cases the R-symmetry index structure is given by the one-loop hamiltonian (2.1), except that the sites it acts on may belong to different chains

$$\tilde{H} : V_i^{(a)} \otimes V_j^{(b)} \to V_k^{(c)} \otimes V_l^{(d)} \qquad \tilde{H} = H\,, \tag{2.24}$$

where a, b, c, d label the chain while i, j, k, l label the sites. The different types of diagrams have a different position dependence. The first type leads to the same position dependence as that of the one-loop corrections to two-point functions, $\ln|x_i - x_j|^2$, while the second type leads to the position dependence typical for one-loop corrections to three-point functions, $\ln|x_i - x_j|^2|x_i - x_k|^2/|x_j - x_k|^2$. Identifying the labels of the positions x_i and the length of the operators L_i, the coefficient of $\ln|x_1 - x_2|^2$ in a generic three-point function is given by the expectation value of

$$\begin{aligned}
&\sum_{s_1=0}^{L_1-1}\sum_{s_2=0}^{L_2-1}\sum_{s_3=0}^{L_3-1}\sum_{\text{all } i,j} \tau(0)_{L_1}^{s_1} \otimes \\
&\otimes \tau(0)_{L_2}^{s_2} H_{i_k i_{k+1}}^{j_k j_{k+1}} |(\phi^{i_1} \dots \phi^{i_k}\phi^{i_{k+1}} \dots \phi^{i_m})(\phi^{i_{m+1}} \dots \phi^{i_{L_1}})\rangle \otimes \\
&\otimes |(\phi^{i_1} \dots \phi^{j_k}\phi^{j_{k+1}} \dots \phi^{i_m})(\phi^{j_{m+1}} \dots \phi^{j_{L_2}})\rangle\langle(\phi_{i_{m+1}} \dots \phi_{i_{L_1}})(\phi_{j_{m+1}} \dots \phi_{j_{L_2}})|\ \tau(0)_{L_3}^{s_3}\,.
\end{aligned} \tag{2.25}$$

Similarly, the coefficient of $\ln|x_3 - x_1|^2|x_3 - x_2|^2/|x_1 - x_2|^2$ is given by the expectation value of

$$\begin{aligned}
&\sum_{s_1=0}^{L_1-1}\sum_{s_2=0}^{L_2-1}\sum_{s_3=0}^{L_3-1}\sum_{\text{all } i,j,i',j'} \tau(0)_{L_1}^{s_1} \otimes \\
&\otimes \tau(0)_{L_2}^{s_2} \left[H_{i_{L_1} j_{m+1}}^{i'_{L_1} j'_{m+1}} I_{i_{m+1} j_{L_2}}^{i'_{m+1} j'_{L_2}} + I_{i_{L_1} j_{m+1}}^{i'_{L_1} j'_{m+1}} H_{i_{m+1} j_{L_2}}^{i'_{m+1} j'_{L_2}} \right] \\
&|(\phi^{i_1} \dots \phi^{i_m})(\phi^{i_{m+1}} \dots \phi^{i_{L_1}})\rangle \otimes |(\phi^{i_1} \dots \phi^{i_m})(\phi^{j_{m+1}} \dots \phi^{j_{L_2}})\rangle \times \\
&\times \langle(\phi_{i'_{m+1}}\phi_{i_{m+2}} \dots \phi_{i_{L_1-1}}\phi_{i'_{L_1}})(\phi_{j'_{m+1}}\phi_{j_{m+2}} \dots \phi_{j_{L_2-1}}\phi_{j'_{L_2}})|\ \tau(0)_{L_3}^{s_3}\,.
\end{aligned} \tag{2.26}$$

The generalization to the other terms appearing in the one-loop corrections to three-point functions is now obvious. For the case of extremal three-point functions[2] there are yet two more contributions due to the fact that the possible Feynman diagrams can have more general topologies (see e.g. [68]). These new contributions are similar to (2.25) and (2.26) and can be easily inferred from Feynman diagrams.

[2]It would be interesting to extend this analysis to higher point functions, where extremality is likely to simplify the calculation. In the context of AdS/CFT extremal and near-extremal n-point functions of chiral operators have been discussed in [66] and [67] respectively.

We will now proceed to describe the calculation of matrix elements of some of the spin chain operators constructed in this section. For the sake of simplicity we concentrate on the SU(2) sector, but some of the formulae extend to larger collections of operators. Though we will mostly be concerned with the representation (2.9) of gauge theory operators on the spin chain, the general expressions which we will review in section 4 are all the necessary ingredients in the calculation of the expectation values of (2.14), (2.20), (2.22), (2.23), (2.25) and (2.26) as well.

3. Algebraic Bethe ansatz: a quick review

The SU(2) sectors of $\mathcal{N}=4$ SYM and some of its deformations are described by the spin 1/2 XXX and XXZ spin chains. The hamiltonian of the XXZ chain is

$$H=\sum_{i=1}^{L}(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\Delta(\sigma_i^z\sigma_{i+1}^z-1))\,, \tag{3.1}$$

where σ_i^a, $a=x,y,z$ are the Pauli matrices acting at the site i and for $\Delta=1$ we recover the XXX chain. This model can be solved by the Bethe Ansatz which we will now briefly review following [62, 69].

The starting point is the R-matrix which satisfies the Yang-Baxter equations. In our cases, the relevant solutions are

$$R(\lambda)=-i\begin{pmatrix}\alpha(\lambda) & 0 & 0 & 0\\ 0 & \beta(\lambda) & \gamma(\lambda) & 0\\ 0 & \gamma(\lambda) & \beta(\lambda) & 0\\ 0 & 0 & 0 & \alpha(\lambda)\end{pmatrix}, \tag{3.2}$$

where λ is the spectral parameter or rapidity, while

$$\alpha(\lambda)=i\frac{\sinh(\lambda+2i\eta)}{\sinh 2i\eta}\,,\qquad \beta(\lambda)=i\frac{\sinh\lambda}{\sinh 2i\eta}\,,\qquad \gamma(\lambda)=i \tag{3.3}$$

for the XXZ chain and

$$\alpha(\lambda)=\lambda+i\,,\qquad \beta(\lambda)=\lambda\,,\qquad \gamma(\lambda)=i \tag{3.4}$$

for the XXX chain. The R-matrix is normalized such that $R(0)$ is the permutation operator.

The monodromy matrix is constructed out of the R matrix as

$$T^{a_L;k_L\ldots k_1}_{b_1;n_L\ldots n_1}(\lambda)=R^{a_Lk_L}_{b_Ln_L}R^{b_Lk_{L-1}}_{b_{L-1}n_{L-1}}\ldots R^{b_2k_1}_{b_1n_1}(\lambda)\equiv\begin{pmatrix}A^{k_L\ldots k_1}_{n_L\ldots n_1}(\lambda) & B^{k_L\ldots k_1}_{n_L\ldots n_1}(\lambda)\\ C^{k_L\ldots k_1}_{n_L\ldots n_1}(\lambda) & D^{k_L\ldots k_1}_{n_L\ldots n_1}(\lambda)\end{pmatrix}, \tag{3.5}$$

where the indices $(k_L,\ldots,k_1)$ and $(n_L,\ldots,n_1)$ correspond to the sites of the chain while a_L and b_1 correspond to an auxiliary space of the same dimension as the dimension of the vector space on the sites of the chain. As a consequence of the Yang-Baxter equations, the monodromy matrix satisfies

$$R(\lambda-\mu)(T(\lambda)\otimes T(\mu))=(T(\mu)\otimes T(\lambda))R(\lambda-\mu)\,, \tag{3.6}$$

where R and $\otimes$ act in the auxiliary space. Examining the various entries of this matrix equation it is relatively easy to find that the off-diagonal entries of T act almost as creation and annihilation operators with respect to the diagonal entries. The relevant matrix elements of (3.6) can be written as

$$
\begin{aligned}
[B(\lambda), B(\mu)] &= [C(\lambda), C(\mu)] = 0\,,\\
[B(\lambda), C(\mu)] &= g(\lambda,\mu)(D(\lambda)A(\mu) - D(\mu)A(\lambda))\,,\\
D(\mu)B(\lambda) &= f(\lambda,\mu)B(\lambda)D(\mu) + g(\mu,\lambda)B(\mu)D(\lambda)\,,\\
A(\mu)B(\lambda) &= f(\mu,\lambda)B(\lambda)A(\mu) + g(\lambda,\mu)B(\mu)A(\lambda)\,,
\end{aligned}
\tag{3.7}
$$

where

$$
f(\lambda,\mu) \equiv f(\lambda-\mu) = \frac{\alpha(\mu-\lambda)}{\beta(\mu-\lambda)}\,, \qquad g(\lambda,\mu) \equiv g(\lambda-\mu) = -\frac{\gamma(\lambda-\mu)}{\beta(\lambda-\mu)}\,. \tag{3.8}
$$

A consequence of the Yang-Baxter equations is that the transfer matrix, which is the trace of the monodromy matrix in the auxiliary space

$$
\tau(\lambda) = \mathrm{Tr}_{aux}\, T(\lambda) = A(\lambda) + D(\lambda)\,, \tag{3.9}
$$

commutes with itself at arbitrary values of spectral parameters. Thus, if the hamiltonian is among the Taylor coefficients of $\tau(\lambda)$, the transfer matrix is the generating functional of an infinite number of mutually commuting conserved charges. The hamiltonian is constructed out of the transfer matrix as

$$
H^{XXZ} = i\sin 2\eta\, \frac{d}{d\lambda} \ln \tau_{XXZ}(\lambda)\Big|_{\lambda=i\eta} - L\cos 2\eta \qquad H^{XXX} = \frac{d}{d\lambda} \ln \tau_{XXX}(\lambda)\Big|_{\lambda=0} - L \tag{3.10}
$$

and $\Delta = \cos\eta$.

To construct the eigenstates of the transfer matrix and the hamiltonian one starts with the (pseudo)vacuum $|0\rangle$ which satisfies

$$
A(\lambda)|0\rangle = \tilde{a}(\lambda)|0\rangle\,, \qquad D(\lambda)|0\rangle = \tilde{d}(\lambda)|0\rangle\,, \qquad C(\lambda)|0\rangle = 0\,, \tag{3.11}
$$

where

$$
\tilde{a}(\lambda) = (-i\alpha(\lambda))^L \quad , \quad \tilde{d}(\lambda) = (-i\beta(\lambda))^L \tag{3.12}
$$

for both the XXX and the XXZ chain. With this starting point the natural ansatz for the eigenvectors of the transfer matrix is

$$
|\Psi_N(\{\lambda\})\rangle = \prod_{i=1}^{N} B(\lambda_i)|0\rangle\,, \qquad N = 0,1,\ldots,L \tag{3.13}
$$

where the arguments λ_i are determined by the requirement that (3.13) be indeed an eivenvector of (3.9). Using the fact that (3.13) is symmetric in λ_i it is possible to find the action

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of the operators $A(\mu)$, $D(\mu)$ and $C(\mu)$ on $|\Psi_N(\{\lambda\})\rangle$

$$\begin{aligned}
A(\mu)\prod_{j=1}^{N}B(\lambda_j)|0\rangle &= \Lambda(\mu)\prod_{j=1}^{N}B(\lambda_j)|0\rangle+\sum_{n=1}^{N}\Lambda_nB(\mu)\prod_{j=1,j\neq n}^{N}B(\lambda_j)|0\rangle\,,\\
D(\mu)\prod_{j=1}^{N}B(\lambda_j)|0\rangle &= \tilde\Lambda(\mu)\prod_{j=1}^{N}B(\lambda_j)|0\rangle+\sum_{n=1}^{N}\tilde\Lambda_nB(\mu)\prod_{j=1,j\neq n}^{N}B(\lambda_j)|0\rangle\,,\\
C(\mu)\prod_{j=1}^{N}B(\lambda_j)|0\rangle &= \sum_{n=1}^{N}M_n\prod_{j=1 j\neq n}^{N}B(\lambda_j)|0\rangle+\sum_{k>n}M_{kn}B(\mu)\prod_{j=1,j\neq k,n}B(\lambda_j)|0\rangle
\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}
\Lambda(\mu) &= a(\mu)\prod_{j=1}^{N}f(\mu,\lambda_j)\,, \qquad \Lambda_n = a(\lambda_n)g(\lambda_n,\mu)\prod_{j=1,j\neq n}f(\lambda_n,\lambda_j)\,,\\
\tilde\Lambda(\mu) &= d(\mu)\prod_{j=1}^{N}f(\lambda_j,\mu)\,, \qquad \tilde\Lambda_n = d(\lambda_n)g(\mu,\lambda_n)\prod_{j=1,j\neq n}f(\lambda_j,\lambda_n)\,,\\
M_n &= g(\mu,\lambda_n)a(\mu)d(\lambda_n)\prod_{j\neq n}f(\lambda_j,\lambda_n)f(\mu,\lambda_j)+g(\lambda_n,\mu)a(\lambda_n)d(\mu)\prod_{j\neq n}f(\lambda_j,\mu)f(\lambda_n,\lambda_j)\,,\\
M_{kn} &= d(\lambda_k)a(\lambda_n)g(\mu,\lambda_k)g(\lambda_n,\mu)f(\lambda_n,\lambda_k)\prod_{j\neq k,n}f(\lambda_j,\lambda_k)f(\lambda_n,\lambda_j)+\\
&\quad +d(\lambda_n)a(\lambda_k)g(\mu,\lambda_n)g(\lambda_k,\mu)f(\lambda_k,\lambda_n)\prod_{j\neq k,n}f(\lambda_j,\lambda_n)f(\lambda_k,\lambda_j)\,.
\end{aligned}\tag{3.15}$$

and

$$a(\lambda)=\tilde a(\lambda-k) \qquad d(\lambda)=\tilde d(\lambda-k)\tag{3.16}$$

with an arbitrary constant k. This shift does not affect any of the functions f and g above because they depend only on differences of rapidities. The standard choices, which make the equations most symmetric, are $k=i/2$ and $k=i\eta$ for the XXX and XXZ chains respectively.

It thus follows that $|\Psi_N(\{\lambda\})\rangle$ is an eigenvector of (3.9) if λ_i satisfy the Bethe equations

$$\prod_{k=1,k\neq j}^{N}\frac{f(\lambda_j,\lambda_k)}{f(\lambda_k,\lambda_j)}=\frac{d(\lambda_j)}{a(\lambda_j)} \qquad (\forall) \quad j=1,\ldots,N\,,\tag{3.17}$$

and the corresponding eigenvalues of the transfer matrix are

$$\tau(\mu)|\Psi_N(\{\lambda\})\rangle=(\Lambda(\mu)+\tilde\Lambda(\mu))|\Psi_N(\{\lambda\})\rangle\,.\tag{3.18}$$

An additional important constraint comes from the requirement that the eigenvectors correspond to gauge invariant operators. This translates into the requirement that they be invariant under shifts along the chain. Using the fact that $R(0)$ is the permutation operator, it follows that $\tau(0)$ generates such shifts. Thus, the last constraint on the eigenvectors is

$$\Lambda\left(\frac{i}{2}\right)+\tilde\Lambda\left(\frac{i}{2}\right)=1\,.\tag{3.19}$$

4. Computation of correlation functions

To compute the various expectation values and scalar products derived in section 2 it is necessary to find a realization of the spin operators in terms of the entries of the monodromy matrix. Such a representation makes it possible to use the relations (3.7) and (3.14) to perform computations.

4.1 Inverse scattering method

The inverse scattering method expresses the local spin operators in terms of the matrix elements of the transfer matrix. The idea of this method is relatively easy to understand. It is based on the fact that there exists a value for the rapidity λ for which the R-matrix becomes the permutation operator. Using its properties we can see that

$$T(0) = P_{0L}P_{0L-1}\dots P_{01} = P_{01}P_{1L}P_{1L-1}\dots P_{12} = P_{01}P_{12}P_{23}\dots P_{L-1L} = P_{01}\,\tau(0)\,. \quad (4.1)$$

Since $P_{01} = \sum_{a=1}^{3}\sigma_0^a\otimes\sigma_1^a$, we find [70] that

$$\begin{aligned} \sigma_1^+ &= C(0)\tau(0)^{L-1} & \sigma_1^- &= B(0)\tau(0)^{L-1} \\ \frac{1}{2}(1+\sigma_1^z) &= A(0)\tau(0)^{L-1} & \frac{1}{2}(1-\sigma_1^z) &= D(0)\tau(0)^{L-1}\,, \end{aligned} \quad (4.2)$$

where we used the fact that translation by the length of the chain acts as the identity operator.[3] From here it is easy to find the spin operators at the site i, using the shift operator $\tau(0)$:

$$O_i = \tau(0)^i O_1 \tau(0)^{-i}\,. \quad (4.3)$$

It is however not hard to see that, using these expressions, the equations (3.14) lead to apparently singular results if we act on some state with two or more operators of the same rapidity. Indeed, both $f(0)$ and $g(0)$ are singular, and going back to equation (3.6) does not yield the necessary commutation relations. Using the explicit form of the spin operators one can easily convince oneself that the scalar products are completely finite. Thus, for computational purposes, it is necessary to regularize (4.2). A convenient regularization is provided by the anisotropic chains, which are based on the same R-matrix as the isotropic one, except that the R-matrices building the monodromy matrix are evaluated at site-dependent shifted rapidities

$$T(\lambda) = R_{0L}(\lambda-\xi_L)R_{0L-1}(\lambda-\xi_{L-1})\dots R_{01}(\lambda-\xi_1)\,, \quad (4.4)$$

and the vacuum state (3.11) and the vacuum energies (3.12) are modified appropriately.

Though the arguments are more involved [71] , the expressions for the spin operators in terms of the entries of the monodromy matrix are quite similar to (4.2) and (4.3):

$$\sigma_i^- = \{\textstyle\prod_{a=1}^{i-1}(A+D)(\xi_a)\}B(\xi_i)\{\prod_{a=i+1}^{L}(A+D)(\xi_a)\}\,,$$

[3] Note that this expression holds for all models based on an R-matrix satisfying $R(0) = P$ and monodromy matrix constructed as in (3.5). Thus, the invariant meaning of the arguments of A, B, C, D and τ in equation (4.2) is the value of the spectral parameter for which the R-matrix becomes the permutation operator. It will occasionally be convenient to shift the rapidity, as in equation (3.16).

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$$\begin{aligned}
\sigma_i^+ &= \{\textstyle\prod_{a=1}^{i-1}(A+D)(\xi_a)\}C(\xi_i)\{\prod_{a=i+1}^{L}(A+D)(\xi_a)\}\,, \\
\frac{1}{2}(1+\sigma_i^z) &= \{\textstyle\prod_{a=1}^{i-1}(A+D)(\xi_a)\}A(\xi_i)\{\prod_{a=i+1}^{L}(A+D)(\xi_a)\}\,, \\
\frac{1}{2}(1-\sigma_i^z) &= \{\textstyle\prod_{a=1}^{i-1}(A+D)(\xi_a)\}D(\xi_i)\{\prod_{a=i+1}^{L}(A+D)(\xi_a)\}\,.
\end{aligned} \tag{4.5}$$

According to [70] similar formulae hold for superalgebras, which generalizes our discussion to some sectors of gauge theory containing fermionic operators.

4.2 Scalar product

As we have seen in section 2, the calculation of correlation functions of gauge invariant operators reduces at the level of the spin chain to the calculation of the scalar products of states constructed out of B and C operators

$$S_N = \langle 0| \prod_{j=1}^{N} C(\lambda_j^C) \prod_{k=1}^{N} B(\lambda_k^B)|0\rangle\,, \tag{4.6}$$

where λ_j are kept arbitrary (i.e. not necessarily satisfying the Bethe equations). There is a large amount of literature devoted to the computation of such inner products, part of it described in detail in [62] . One of the main results is that (4.6) is given by

$$\begin{aligned}
S_N = \prod_{j<k} g(\lambda_j^C,\lambda_k^C) g(\lambda_k^B,\lambda_j^B) \sum \mathrm{sign}(P_C)\mathrm{sign}(P_B) \prod_{j,k} h(\lambda_j^{AB},\lambda_k^{DC}) \prod_{l,m} h(\lambda_l^{AC},\lambda_m^{DB}) \times \\
\times \prod_{l,k} h(\lambda_l^{AC},\lambda_k^{DC}) \prod_{j,m} h(\lambda_j^{AB},\lambda_m^{DB}) \det(M_{DC}^{AB}) \det(M_{DB}^{AC})\,,
\end{aligned} \tag{4.7}$$

where P_C and P_B are the permutations $\{\lambda_1^{AC},\ldots,\lambda_n^{AC},\lambda_1^{DC},\ldots,\lambda_{N-n}^{DC}\}$ of $\{\lambda_1^C,\ldots,\lambda_N^C\}$ and $\{\lambda_1^{DB},\ldots,\lambda_n^{DB},\lambda_1^{AB},\ldots,\lambda_{N-n}^{AB}\}$ of $\{\lambda_1^B,\ldots,\lambda_N^B\}$ respectively, and

$$(M_{DC}^{AB})_{jk} = t(\lambda_j^{AB},\lambda_k^{DC}) d(\lambda_k^{DC}) a(\lambda_j^{AB})\,, \qquad t(\lambda,\mu) = \frac{g(\lambda,\mu)^2}{f(\lambda,\mu)}\,, \qquad h(\mu,\lambda) = \frac{f(\mu,\lambda)}{g(\mu,\lambda)}\,. \tag{4.8}$$

This result is derived using the commutation relations (3.7) as well as various properties of rational functions. When the in-state is an energy eigenstate (i.e. $\{\lambda^B\}$ is a solution of the Bethe equations (3.17)) (4.6) simplifies considerably and takes the following form [71]

$$S_N(\{\lambda^C\},\{\lambda^B\}) = \frac{\prod_i d(\lambda_i^C) \prod_a d(\lambda_a^B) \det H(\{\lambda_i^C\},\{\lambda_k^B\})}{\prod_{j>k} \varphi(\lambda_k^C - \lambda_j^C) \prod_{l<m} \varphi(\lambda_m^B - \lambda_l^B)} \tag{4.9}$$

where

$$H_{ab} = \frac{\varphi(\eta)}{\varphi(\lambda_a^B - \lambda_b^C)} \left[\frac{a(\lambda_b^C)}{d(\lambda_b^C)} \prod_{m\neq a} \varphi(\lambda_m^B - \lambda_b^C + \eta_0) - \prod_{m\neq a} \varphi(\lambda_m^B - \lambda_b^C - \eta_0) \right]. \tag{4.10}$$

The function $\varphi(\lambda)$ equals λ and $\sinh\lambda$ for the XXX and XXZ chains, respectively, while η_0 equals i and $2i\eta$ for the XXX and XXZ chains, respectively.

The entries of the matrix H have potential singularities if some rapidity λ_k^B of the out-state approaches one of the rapidities λ_m^C of the in-state. This apparent problem is however cured by the fact that, in this limit, the square bracket in (4.10) turns out to vanish due to the Bethe equations (3.17). This type of scalar product is sufficient for the calculation of three-point function coefficients and will be useful in the examples we will discuss in section 5. In the limit in which the two states become identical, (4.9) becomes the square of the norm of the Bethe eigenstates given by the Gaudin formula proven in [72]

$$\langle 0| \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k)|0\rangle = \prod_i d(\lambda_i) d(\lambda_i) \prod_{a\neq b} f(\lambda_a, \lambda_b) \det \Phi'(\{\lambda\}) \,, \tag{4.11}$$

where Φ' is an $N \times N$ matrix with the elements

$$\Phi'_{ab} = -\frac{\partial}{\partial \lambda_b} \log \left(\frac{a(\lambda_a)}{d(\lambda_a)} \prod_{k=1, k\neq a}^{N} \frac{f(\lambda_a, \lambda_k)}{f(\lambda_k, \lambda_a)} \right) . \tag{4.12}$$

The explicit form of these matrix elements is not too complicated and simplifies even more in the thermodynamic limit ($N, L \to \infty$ with $\frac{N}{L}$ fixed) as well as in the long chain limit ($L \to \infty$ with N fixed). The explicit form of (4.12) is

$$\Phi'_{ab} = -\delta_{ab} \left[\frac{a'}{a}(\lambda_a) - \frac{d'}{d}(\lambda_a) + \sum_{k\neq a} K(\lambda_a - \lambda_k) \right] + K(\lambda_a - \lambda_b) \tag{4.13}$$

where

$$K(\lambda) = \frac{f'}{f}(\lambda) + \frac{f'}{f}(-\lambda) \,. \tag{4.14}$$

It turns out that K is the kernel of the integral equation for the density of rapidities in the thermodynamic limit. After using the Bethe equations in the same limit, the coefficient of the Kronecker δ becomes the density of rapidities. Combining everything, Φ' is

$$\Phi'_{ab}(\{\lambda\}) = L \left[\delta_{ab} \rho(\lambda_a) + \frac{1}{L} K(\lambda_a - \lambda_b) \right] . \tag{4.15}$$

The ratio N/L enters here through the normalization of ρ.

In the long chain limit the result is even simpler because the solutions of the Bethe equations are of the order of the length of the chain, $\lambda_a = L/2\pi n_a$. Thus all terms depending on K in (4.13) have subleading contributions and it is not necessary to solve the equation for ρ. The matrix Φ' becomes diagonal, with entries given by the first two terms in (4.13) and its determinant is therefore

$$\det \Phi'(\{\lambda\}) = \prod_{a=1}^{N} \left[-\frac{a'}{a}(\lambda_a) + \frac{d'}{d}(\lambda_a) \right] . \tag{4.16}$$

Each term in the product is of order of the inverse length of the chain. At first sight, this dependence as well as the extra factors in (4.11) seem to be in contradiction with the known results concerning long operators. Let us therefore briefly discuss the relation between Bethe eigenstates and gauge theory operators.

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Since we picked a normalized vacuum state $|0\rangle$, the corresponding gauge theory operator must also be normalized. We therefore have

$$|0\rangle \quad \longleftrightarrow \quad \frac{1}{\sqrt{L}}\,\mathrm{Tr}[(\phi^1)^L]\,. \tag{4.17}$$

If creation operators are present, the solution to the apparent puzzle implied by (4.16) is provided by the cyclicity of the trace and the fact that the action of the creation operators $B(\lambda)$ brings in additional rapidity-dependent factors. The relation between (3.13) and the non-normalized gauge theory operators is, roughly,

$$|\Psi_N(\{\lambda\})\rangle = \prod_{i=1}^{N} B(\lambda_i)|0\rangle \sim \left[\prod_{i=1}^{N} d(\lambda_i)\right]\left[\prod_{i=1}^{N}\frac{\gamma(\lambda_i)}{\beta(\lambda_i)}\right]\frac{\mathcal{O}^{\mathrm{non-norm}}}{\sqrt{L}}\,, \tag{4.18}$$

where $\mathcal{O}^{\mathrm{non-norm}}$ is, as usual, a linear combination of traces of products of $(L-N)$ ϕ^1 and N ϕ^2 fields while $\sqrt{L}$ in the denominator comes from the normalization of the vacuum state (4.17) . Thus, $[\prod_{i=1}^{N} d(\lambda_i)]$ cancels the similar factor in the norm (4.11) of $|\Psi_N(\{\lambda\})\rangle$, while the other factor contributes L^{-N}. Thus, combining (4.18) and (4.16) we find that the normalized BMN operators are related to the non-normalized ones as

$$\mathcal{O}^{\mathrm{norm}} \sim \frac{1}{L^{(N+1)/2}}\mathcal{O}^{\mathrm{non-norm}}\,. \tag{4.19}$$

Using the cyclicity of the trace we can always place a ϕ^2 field as the first field inside the trace; this reduces the number of terms in $\mathcal{O}^{\mathrm{non-norm}}$ by a factor of L and simply multiplies the resulting operator by L. We therefore find

$$|\Psi_N(\{\lambda\})\rangle \quad \longleftrightarrow \quad \mathcal{O}^{\mathrm{norm}} \sim \frac{\mathcal{O}_0^{\mathrm{non-norm}}}{L^{(N-1)/2}}\,, \qquad \mathcal{O}_0^{\mathrm{non-norm}} = \sum_p \mathrm{Tr}[\phi^2 K_p(\phi^1,\phi^2,\{\lambda\})]\,, \tag{4.20}$$

which is in agreement with the known gauge theory results. Here K_p denotes some monomial in the fields with a λ-dependent coefficient. It is certainly possible to do a more detailed analysis and keep also the subleading terms. We will see this explicitly in an example in the next section.

5. Examples

Let us illustrate the method of computing correlation functions with some examples. All examples will involve states created by two creation operators, so we first normalize them. For a chain of length L the corresponding solution of the Bethe equations (3.17) can be explicitly found for the XXX chain

$$\lambda_1 = -\lambda_2 = -\frac{1}{2}\mathrm{ctg}\frac{\pi n}{L-1}\,, \tag{5.1}$$

where λ_i are the shifted rapidities (3.16) . Then equation (4.11) implies that the normalized state is

$$\begin{aligned}|\Psi_2^{\mathrm{norm}}(n)\rangle &= \mathcal{N}_2 B(\lambda_1)B(-\lambda_1)|0\rangle \\ \mathcal{N}_2 &= \frac{\lambda_1}{\sqrt{L(L-1)}}\left(\lambda_1^2+\frac{1}{4}\right)^{\frac{1}{2}-L} = \frac{4^{L-1}}{\sqrt{L(L-1)}}\cos\frac{\pi n}{L-1}\sin^{2(L-1)}\frac{\pi n}{L-1}\,. \end{aligned} \tag{5.2}$$

Extracting the operators B out of the monodromy matrix and acting with them on the vacuum state leads to an explicit expression for the (normalized) gauge theory operator associated to the Bethe state $|\Psi_2^{\rm norm}(n)\rangle$

$$\mathcal{O}_n = \frac{1}{\sqrt{L-1}} \sum_{s=0}^{L-2} \cos\frac{\pi n(2s+1)}{L-1}\; \mathrm{Tr}[\phi^2(\phi^1)^s\phi^2(\phi^1)^{L-2-s}]\,, \tag{5.3}$$

in agreement with known results [73] . For other spin chains some simple normalized gauge theory operators corresponding to Bethe states have been constructed in [14]. Calculations similar to the one leading to (5.3) yield (after solving the Bethe equations) all finite length operators in the SU(2) sector with definite one-loop anomalous dimensions.

Example one. A simple example of the type in which two operators have the same length is provided by a three-point function of a generator of the Cartan subalgebra of the R-symmetry group

$$\mathrm{Tr}[\phi^1\bar\phi_1 - \phi^2\bar\phi_2] \tag{5.4}$$

and two other operators. According to the discussion in section 2, the tree-level coefficient of this correlation function is given by the following expectation value:

$$\langle\Psi_N^{\rm norm}(\{\mu\})|\sum_{i=1}^{L}\sigma_i^z|\Psi_N^{\rm norm}(\{\lambda\})\rangle = L\,\langle\Psi_N^{\rm norm}(\{\mu\})|\sigma_1^z|\Psi_N^{\rm norm}(\{\lambda\})\rangle\,. \tag{5.5}$$

Using the solution (4.2) of the inverse scattering problem this matrix element reduces to the matrix element of $A(0)-D(0)$ between the in- and out-states above. In this computation it is not necessary to use the regularization provided by the inhomogeneous chain, because there are no potential singularities from contractions. This matrix element was computed in [71] and the result is the one expected on physical grounds

$$\langle\Psi_N^{\rm norm}(\{\mu\})|\sum_{i=1}^{L}\sigma_i^z|\Psi_N^{\rm norm}(\{\lambda\})\rangle = (L-2N)\delta(\{\lambda\}-\{\mu\})\,, \tag{5.6}$$

where $\delta(\{\lambda\}-\{\mu\})$ vanishes unless the set $\{\lambda\}$ is the same as the set $\{\mu\}$. Indeed, the operator (5.4) counts the difference between the number of ϕ^1 and ϕ^2 fields. Since the action of B replaces ϕ^1 with ϕ^2, the result should be the difference between the number of ϕ^1 fields in the vacuum and twice the number of creation operators. Furthermore, since the hamiltonian commutes with (5.4) , the latter does not mix the energy eigenstates, which explains the delta function. For the computation of correlators with more than one σ_i^z see [74]–[75].

Example two. A more complicated example is the extremal three-point function

$$\langle\mathrm{Tr}[\bar\phi_1^{L-2}](x)\,\mathrm{Tr}[\bar\phi_2^2](y)\mathcal{O}_2(z)\rangle\,, \tag{5.7}$$

where $\mathcal{O}_2$ is the normalized operator corresponding to $|\Psi_2^{\rm norm}(n)\rangle$. Using equation (2.14) it is not hard to find that the tree level normalization of (5.7) is

$$c_0 = L\sum_{i_1\ldots i_{L-2}=1}^{2} {}_{L-2}\langle 0|\phi^{i_1}\ldots\phi^{i_{L-2}}\rangle\langle\bar\phi_2\bar\phi_2\bar\phi_{i_1}\ldots\bar\phi_{i_{L-2}}|\Psi_2^{\rm norm}(n)\rangle_L\,, \tag{5.8}$$

where the intermediate states are considered to be normalized,[4] the indices on the vacuum states represent the length of the chain describing them and the overall factor of L comes from the sum over the powers of the shift operator in (2.14). The scalar product between the out vacuum state and the intermediate state forces the intermediate state to also be the vacuum state $|0\rangle_{L-2}$. Thus, only one term in the sum survives

$$c_0 = L\mathcal{N}_2\langle\bar{\phi}_2\bar{\phi}_2\bar{\phi}_1\ldots\bar{\phi}_1|B(\lambda_1)B(-\lambda_1)|0\rangle_L\,, \tag{5.9}$$

where $\mathcal{N}_2$ is the normalization constant of the in-state (5.2). Using the solution (4.2) of the inverse scattering problem and the fact that (5.1) represents shifted rapidities, the remaining scalar product reduces to[5]

$$c_0 = L\mathcal{N}_2\langle 0|\,C(\tfrac{i}{2})C(\tfrac{i}{2})\,B(\lambda_1)B(-\lambda_1)|0\rangle\,. \tag{5.10}$$

Using (3.14) and the regularization provided by the anisotropic chain or the general expression for the scalar product between an energy eigenstate and an arbitrary state (4.9), we find

$$\langle 0|C(\frac{i}{2})C(\frac{i}{2})B(\lambda_1)B(-\lambda_1)|0\rangle = 2\left(\lambda_1^2+\frac{1}{4}\right)^{L-1} = 2^{3-2L}\sin^{2(1-L)}\frac{\pi n}{L-1}\,. \tag{5.11}$$

Combining this with the normalization of the in-state (5.2) leads to the three-point function coefficient

$$c_0 = \sqrt{\frac{L}{L-1}}\frac{4\lambda_1}{\sqrt{1+4\lambda_1^2}} = 2\sqrt{\frac{L}{L-1}}\cos\frac{\pi n}{L-1}\,. \tag{5.12}$$

Similar coefficients have been computed [76] for some operators dual to string modes in plane wave background. The equation (5.12) is more general since it is valid for operators of arbitrary length.

As a side remark, it is not hard to see using the analysis of section 2 that the same tree-level three-point function coefficient arises, up to some numerical coefficient, if we replace $\mathrm{Tr}[\bar{\phi}_2^2]$ with $\mathrm{Tr}[(\phi^1)^k\bar{\phi}_2^2]$ for any $k \leq L-4$ and also appropriately adjust the length of the chain describing the out-state.

Using the analog of the equations (2.25) and (2.26) for the SU(2) sector it is not hard to find the coefficients of the one-loop corrections to (5.7). The relevant gauge theory Feynman diagrams, grouped as discussed in section 2, are listed in figure 3. The white dots correspond to ϕ_1 while black dots correspond to ϕ_2. Some of the contributions are particularly simple. Since both $\mathrm{Tr}[\bar{\phi}_1^{L-2}]$ and $\mathrm{Tr}[\bar{\phi}_2^2]$ are BPS operators, they are annihilated by the hamiltonian of the spin chain and the contribution of the analog of equation (2.25) (first two diagrams in figure 3) vanishes. The contribution of the analog of (2.26) does not vanish and it is a sum of two terms. The simplest contribution (third diagram in figure 3) comes from the identity operator in the hamiltonian and contributes c_0 to the coefficient

[4] In fact, it is not hard to see from section 2 that the intermediate states are already unit normalized.

[5] As discussed before, the arguments of the C operators above must be the values of the rapidities for which the R-matrix becomes the permutation operator. For the shifted rapidities this is $\lambda_0 = \frac{i}{2}$.

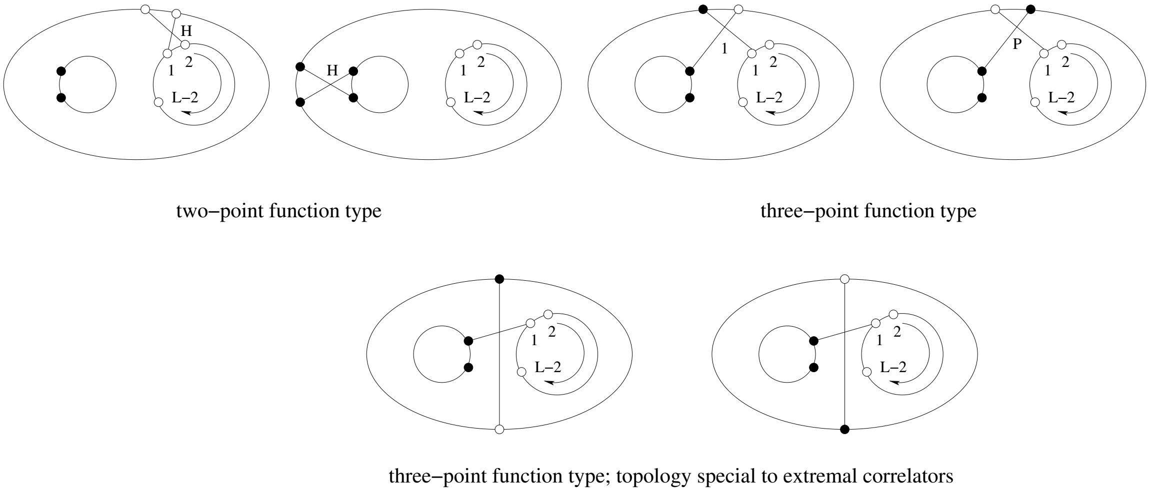

Figure 3: Various diagrams contributing to (5.7) at one-loop level.

of $\ln|z-x|^2|z-y|^2/|x-y|^2$. A similar contribution comes from the interactions typical to extremal three-point functions (fifth and sixth diagrams above).

The final contribution (fourth diagram in figure 3) comes from the permutation operator and is proportional to

$$C_P^{(1)} = \langle 0|C(\tfrac{i}{2})(A(\tfrac{i}{2})+D(\tfrac{i}{2}))C(\tfrac{i}{2})|\Psi_2^{\rm norm}(n)\rangle\,. \tag{5.13}$$

This matrix element is a special case of a spin-spin correlation function and measures the probability of having two down spins with an up spin between them. The computation of this scalar product is relatively easy. As before, we need to regularize it by shifting the arguments of the operators in (5.13) at arbitrary positions. It is not hard to see that the nontrivial normalization factor of the regularized state is irrelevant, so we will not compute it explicitly. Defining[6]

$$\begin{aligned} S(\xi) &= \langle 0|C(\xi)C(\tfrac{i}{2})|\Psi_2^{\rm norm}(n)\rangle \\ &= \frac{\lambda_1\sqrt{1+4\lambda_1^2}\,(\frac{1}{2}-i\xi)}{\sqrt{L(L-1)}(\lambda_1^2-\xi^2)}\left[\left(\frac{1}{2}-i\xi\right)^{L-1}-\left(-\frac{1}{2}-i\xi\right)^{L-1}\right] \end{aligned} \tag{5.14}$$

which can be easily computed using (4.9) , the scalar product (5.13) is given by

$$C_P^{(1)} = \left(a\left(\frac{i}{2}\right)+d\left(\frac{i}{2}\right)\right)S(\frac{i}{2})+i\left[a(\xi)^2\partial_\xi\left(\frac{S(\xi)}{a(\xi)}\right)-d(\xi)^2\partial_\xi\left(\frac{S(\xi)}{d(\xi)}\right)\right]\Bigg|_{\xi=\frac{i}{2}}\,. \tag{5.15}$$

Using the explicit form of a, d and $S(\xi)$ we find that the last one-loop three-point function coefficient is given by

$$C_P^{(1)} = c_0\left[1-4\sin^2\frac{\pi n}{L-1}\right] = \sqrt{\frac{L}{L-1}}\cos\frac{3\pi n}{L-1}\,. \tag{5.16}$$

Collecting the contributions of all diagrams in figure 3 yields the one-loop correction to the three-point function (5.7).

[6]Clearly, $S(\frac{i}{2}) = c_0$.

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